

Catalogue of Spacetimes

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Morris-Thorne metric

The Morris-Thorne wormhole metric with proper radial coordinates (t, l, θ, ϕ) is given by the line element

$$ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

where b_0 is the throat size and c is the speed of light.

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Coordinates and metric:

■ Clearing the values, setting the dimension and defining a list of coordinates

```
Clear[coord, metric, inversemetric, affine, t, r, \theta, \phi]

n := 4

coord := {t, l, \theta, \phi}
```

■ Metric and inverse Metric

The metric $g_{\mu\nu}$ is given by the list

```
metric := {{-c^2, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, b0^2 + l^2, 0}, {0, 0, 0, (b0^2 + l^2) Sin[\theta]^2}}
```

and the inverse metric $g^{\mu\nu}$ follows from

```
inversemetric := Simplify[Inverse[metric]]
```

■ Christoffel symbols of the second kind

The Christoffel symbols of the second kind are defined as $\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\rho}(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho})$

```
affine := affine = Simplify[Table[(1/2)
  Sum[inversemetric[[\mu, \rho]] (D[metric[[\rho, \nu]], coord[[\lambda]] + D[metric[[\rho, \lambda]], coord[[\nu]]] -
    D[metric[[\nu, \lambda]], coord[[\mu]])), {\rho, 1, n}, {\nu, 1, n}, {\lambda, 1, n}, {\mu, 1, n}]]

listaffine := Table[If[UnsameQ[affine[[\nu, \lambda, \mu]], 0],
  {Style[Subsuperscript[\Gamma, Row[{coord[[\nu]], coord[[\lambda]]}], coord[[\mu]], 18],
  "=", Style[affine[[\nu, \lambda, \mu]], 14]}], {\lambda, 1, n}, {\nu, 1, n}, {\mu, 1, n}]
```

```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 3], TableSpacing -> {1, 2}]
```

$$\Gamma_{1\theta}^{\theta} = \frac{1}{b0^2+1^2}$$

$$\Gamma_{\theta\theta}^1 = -1$$

$$\Gamma_{1\phi}^{\phi} = \frac{1}{b0^2+1^2}$$

$$\Gamma_{\theta\phi}^{\phi} = \text{Cot}[\theta]$$

$$\Gamma_{\phi\phi}^1 = -1 \text{Sin}[\theta]^2$$

$$\Gamma_{\phi\phi}^{\theta} = -\text{Cos}[\theta] \text{Sin}[\theta]$$

■ Riemann tensor

The Riemann tensor is given by means of the Christoffel symbols

$$R^{\mu}_{\nu\rho\sigma} = \Gamma_{\nu\sigma,\rho}^{\mu} - \Gamma_{\nu\rho,\sigma}^{\mu} + \Gamma_{\rho\lambda}^{\mu} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\sigma\lambda}^{\mu} \Gamma_{\nu\rho}^{\lambda}$$

```
riemann := riemann = Table[D[affine[[v, sigma, mu]], coord[[rho]] - D[affine[[v, rho, mu]], coord[[sigma]] +
  Sum[affine[[rho, lambda, mu]] affine[[v, sigma, lambda]] - affine[[sigma, lambda, mu]] affine[[v, rho, lambda]], {lambda, 1, n}],
  {mu, 1, n}, {nu, 1, n}, {rho, 1, n}, {sigma, 1, n}]
```

The Riemann tensor with lower indices reads $R_{\mu\nu\rho\sigma} = g_{\mu\kappa} R^{\kappa}_{\nu\rho\sigma}$

```
riemannDn := riemannDn = Table[Simplify[Sum[metric[[mu, kappa]] riemann[[kappa, nu, rho, sigma]], {kappa, 1, n}],
  {mu, 1, n}, {nu, 1, n}, {rho, 1, n}, {sigma, 1, n}]
```

```
listRiemann := Table[If[UnsameQ[riemannDn[[mu, nu, rho, sigma]], 0],
  {Style[Subscript[R, Row[{coord[[mu]], coord[[nu]], coord[[rho]], coord[[sigma]]}], 16],
  "=", riemannDn[[mu, nu, rho, sigma]]}], {nu, 1, n}, {mu, 1, n}, {rho, 1, n}, {sigma, 1, n}]
```

```
TableForm[Partition[DeleteCases[Flatten[listRiemann], Null], 3], TableSpacing -> {2, 2}]
```

$$R_{1\theta 1\theta} = -\frac{b0^2}{b0^2+1^2}$$

$$R_{1\phi 1\phi} = -\frac{b0^2 \text{Sin}[\theta]^2}{b0^2+1^2}$$

$$R_{\theta\phi\theta\phi} = b0^2 \text{Sin}[\theta]^2$$

■ Ricci tensor

The Ricci tensor follows from the contraction of the Riemann tensor: $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$

```
ricci := ricci = Table[Simplify[Sum[riemann[[rho, mu, rho, nu]], {rho, 1, n}], {mu, 1, n}, {nu, 1, n}]
```

```
listRicci :=
  Table[If[UnsameQ[ricci[[mu, nu]], 0], {Style[Subscript[R, Row[{coord[[mu]], coord[[nu]]}], 16],
  "=", Style[ricci[[mu, nu]], 16]}], {nu, 1, 4}, {mu, 1, 4}]
```

```
TableForm[Partition[DeleteCases[Flatten[listRicci], Null], 3], TableSpacing -> {1, 2}]
```

$$R_{11} = -\frac{2 b0^2}{(b0^2+1^2)^2}$$

Ricci scalar

The Ricci scalar is the given by the contraction of the Ricci tensor $R = R^\alpha_\alpha$

```
ricciscalar :=
  ricciscalar = Simplify[Sum[Sum[inversemetric[[μ, ν]] ricci[[ν, μ]], {μ, 1, n}], {ν, 1, n}]]
```

```
ricciscalar
```

$$-\frac{2 b_0^2}{(b_0^2 + 1^2)^2}$$

■ Kretschman scalar

The Kretschman scalar is defined as $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = R^{\alpha\beta}_{\gamma\delta} R^{\gamma\delta}_{\alpha\beta}$

```
riemannUp :=
  riemannUp = Table[Simplify[Sum[inversemetric[[ν, κ]] riemann[[μ, κ, ρ, σ]], {κ, 1, n}],
    {μ, 1, n}, {ν, 1, n}, {ρ, 1, n}, {σ, 1, n}]
```

```
kretschman := Simplify[
  Sum[Sum[Sum[Sum[riemannUp[[μ, ν, ρ, σ]] riemannUp[[ρ, σ, μ, ν]], {μ, 1, n}], {ν, 1, n}],
    {ρ, 1, n}], {σ, 1, n}]]
```

```
kretschman
```

$$\frac{12 b_0^4}{(b_0^2 + 1^2)^4}$$